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SOME PROBLEMS OF THE STABILITY OF A REVERSIBLE SYSTEM WITH A SMALL PARAMETER†

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A solution of the problem of the stability of periodic and conditionally periodic nearly autonomous systems, and a qualitative analysis of systems close to resonance are given for reversible systems using a single approach. In addition, the problem of the rolling of Chaplygin's nearly dynamically symmetrical sphere is considered. It is shown that rolling along a straight line is formally stable for all values of the parameters. For a nearly homogeneous sphere a main resonance is possible which leads to instability.

1. THE QUASI-AUTONOMOUS REVERSIBLE SYSTEM

Consider the problem of the stability of the zero solution of the system

$$\begin{aligned}
&\mathbf{x}^{\cdot} = \mathbf{X}_0(\mathbf{x}) + \varepsilon\mathbf{X}_1(\varepsilon, \mathbf{x}, t), \qquad \mathbf{x} \in \mathbf{R}^p \\
&\mathbf{X}_0(\mathbf{0}) = \mathbf{0}, \quad \mathbf{X}_1(\varepsilon, \mathbf{0}, t) \equiv \mathbf{0}, \quad \mathbf{X}_1(\varepsilon, \mathbf{x}, t + 2\pi) = \mathbf{X}_1(\varepsilon, \mathbf{x}, t) \\
&\mathbf{M}[\mathbf{X}_0(\mathbf{x}) + \varepsilon\mathbf{X}_1(\varepsilon, \mathbf{x}, t)] + \mathbf{X}_0(\mathbf{M}\mathbf{x}) + \varepsilon\mathbf{X}_1(\varepsilon, \mathbf{M}\mathbf{x}, -t) \equiv \mathbf{0}
\end{aligned} \tag{1.1}$$

which is reversible with respect to a mapping $\mathbf{M}$ of the phase space. Here ε is a small parameter and $\mathbf{M}$ is a certain constant non-degenerate matrix. We shall assume that the right-hand sides are analytic in the variable $\mathbf{x}$ and in the small parameter ε, and can be represented by Taylor–Fourier series.

For simplicity, we will investigate the case of involution when $\mathbf{M}^2 = \mathbf{E}$, and the canonical form of the matrix $\mathbf{M}$ is

$$\mathbf{M} = \begin{Vmatrix} \mathbf{E}_l & \mathbf{0} \\ \mathbf{0} & -\mathbf{E}_n \end{Vmatrix} \quad (l + n = p)$$

($\mathbf{E}_j$ is the identity matrix of order j). Such a situation occurs in mechanical systems where, as a rule $l \geqslant n$. Hence the characteristic equation of the autonomous system obtained from (1.1) at $\varepsilon = 0$ has not less than $m = n - l$ simple zero roots [1]. The remaining roots are divided into pairs $\pm\lambda_s$ $(s = 1, \ldots, n)$, and stability is therefore possible only when all $\lambda_s^2 \leqslant 0$. In the subsequent discussion we shall assume that $\lambda_s^2 < 0$ $(s = 1, \ldots, n)$, and the simple elementary divisors correspond to multiple λ_s.

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With these assumptions, system (1.1) can be reduced to the form

$$\begin{aligned}
\boldsymbol{\xi}^{\cdot} &= \boldsymbol{\Xi}_0(\boldsymbol{\xi}, \boldsymbol{\eta}, \overline{\boldsymbol{\eta}}) + \varepsilon \boldsymbol{\Xi}_1(\varepsilon, \boldsymbol{\xi}, \boldsymbol{\eta}, \overline{\boldsymbol{\eta}}, t) \\
\boldsymbol{\eta}^{\cdot} &= \boldsymbol{\Lambda}\boldsymbol{\eta} + \mathbf{H}_0(\boldsymbol{\xi}, \boldsymbol{\eta}, \overline{\boldsymbol{\eta}}) + \varepsilon \mathbf{H}_1(\varepsilon, \boldsymbol{\xi}, \boldsymbol{\eta}, \overline{\boldsymbol{\eta}}, t) \\
\overline{\boldsymbol{\eta}}^{\cdot} &= -\boldsymbol{\Lambda}\overline{\boldsymbol{\eta}} + \overline{\mathbf{H}}_0(\boldsymbol{\xi}, \boldsymbol{\eta}, \overline{\boldsymbol{\eta}}) + \varepsilon \overline{\mathbf{H}}_1(\varepsilon, \boldsymbol{\xi}, \boldsymbol{\eta}, \overline{\boldsymbol{\eta}}, t) \\
\boldsymbol{\Lambda} &= \operatorname{diag}(\lambda_1, \ldots, \lambda_n)
\end{aligned} \tag{1.2}$$

by means of a linear transformation independent of ε. Here $\boldsymbol{\xi}$ is a real m-vector, and $\boldsymbol{\eta}$ and $\overline{\boldsymbol{\eta}}$ are complex-conjugate n-vectors. The linear part of the autonomous system is written explicitly and the bar denotes the complex conjugate. Here, the right-hand sides of (1.2) contain [1–3] only pure imaginary coefficients by virtue of the reversibility.

Henceforth, when examining problems with small parameters we shall use the approach employed in [2], treating ε as the local variable, which satisfies the equation $\dot{\varepsilon} = 0$, with subsequent application of the normalizing procedure to the system of equations obtained.

The linear system. We will first consider the problem of calculating the characteristic exponents of the system linearized with respect to the variables $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ and $\overline{\boldsymbol{\eta}}$. This problem has been considered in detail in [2]. A very simple and comprehensive solution for a reversible system, which also holds for a conditionally periodic right-hand side in (1.1), is obtained below.

We will normalize the system linearly with respect to $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ and $\overline{\boldsymbol{\eta}}$, but non-linearly with respect to all the variables ε, $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ and $\overline{\boldsymbol{\eta}}$. Then, in normal form obtained

$$\begin{aligned}
\varepsilon^{\cdot} &= 0 \\
\xi_j^{\cdot} &= \xi_j \Sigma g_{j\mu\nu q r} \varepsilon^{\mu} \boldsymbol{\xi}^{\nu} \boldsymbol{\eta}^{q} \overline{\boldsymbol{\eta}}^{r} e^{ipt} \quad (j = 1, \ldots, m) \\
\eta_s^{\cdot} &= \lambda_s \eta_s + \eta_s \Sigma h_{s\mu\nu q r} \varepsilon^{\mu} \boldsymbol{\xi}^{\nu} \boldsymbol{\eta}^{q} \overline{\boldsymbol{\eta}}^{r} e^{ipt} \\
\overline{\eta}_s^{\cdot} &= -\lambda_s \overline{\eta}_s + \overline{\eta}_s \Sigma \overline{h}_{s\mu\nu q r} \varepsilon^{\mu} \boldsymbol{\xi}^{\nu} \overline{\boldsymbol{\eta}}^{q} \boldsymbol{\eta}^{r} e^{-ipt} \quad (s = 1, \ldots, n)
\end{aligned} \tag{1.3}$$

only the resonance terms are non-zero, for which [2]

$$\begin{aligned}
&(q_1 - r_1)\lambda_1 + \ldots + (q_n - r_n)\lambda_n + ip = 0 \\
&\nu_1 + \ldots + \nu_m + q_1 + \ldots + q_n + r_1 + \ldots + r_n = 0
\end{aligned} \tag{1.4}$$

The exponents ν_j, q_s and r_k are non-negative integers (p is integer), except ν_j for the jth equation in $\boldsymbol{\xi}$ and q_s for the sth equation in $\boldsymbol{\eta}$. Here ν_j and q_s can also take the value -1.

We shall clarify the structure of the normal form (1.3). If we have $\nu_j = 0$ in the equation for ξ_j then all the remaining exponents ν_k $(k \neq j)$, q_s, r_α are equal to zero, as follows from (1.4). The corresponding constant coefficient of ξ_j is then zero by virtue of the invariance of system (1.3) to the replacement of $(t, \boldsymbol{\xi}, \boldsymbol{\eta}, \overline{\boldsymbol{\eta}})$ and $(-t, \boldsymbol{\xi}, \overline{\boldsymbol{\eta}}, \boldsymbol{\eta})$. When $\nu_j = -1$ the second relation of (1.4) is satisfied for one of ν_k $(k \neq j)$, q_s, r_α equal to 1. If $\nu_k = 1$, as a consequence of the reversibility, the coefficient of ξ_k equals zero. If $q_s = 1$ then the first relation of (1.4) implies $\lambda_s + ip = 0$, and hence there is simultaneously the term with $r_s = 1$ $(-\lambda_s - ip = 0)$.

Now consider the equation for η_s. An analysis of conditions (1.4) shows that only the coefficients containing time explicitly, for which $q_s = 1$, are non-zero here. Then, either $r_s = 1$ and the resonance $2\lambda_s + ip$ occurs, or $r_j = 1$ $(j \neq s)$ and the resonance $\lambda_s + \lambda_j = ip$ occurs, or $q_j = 1$ and the resonance $\lambda_s - \lambda_j = ip$ occurs.

Hence, if $i(\lambda_s \pm \lambda_j)$ is not an integer, the normal form (1.3) has the form

$$\boldsymbol{\xi}^{\cdot} = \mathbf{0}, \quad \eta_s^{\cdot} = \lambda_s^*(\varepsilon)\eta_s, \quad \overline{\eta}_s^{\cdot} = -\lambda_s^*(\varepsilon)\overline{\eta}_s \quad (s = 1, \ldots, n)$$

where

$$\lambda_s^* = \lambda_s + \varepsilon\lambda_s^{(1)} + \varepsilon^2\lambda_s^{(2)} + \ldots \quad (\lambda_s^{(j)} = \text{const})$$

are series in ε with purely imaginary coefficients. From the convergence of the normalizing transformation in this case [2], we conclude that the following theorem holds.

Theorem 1. If $\lambda_s \pm \lambda_j$ is not an integer i, all the characteristic exponents are pure imaginary and they are calculated as the roots of the characteristic equation of the normalized system (1.3).

Corollaries. 1. When the conditions of Theorem 1 are satisfied the characteristic exponents to a first approximation in ε may be obtained by replacing, in the linear system, the periodic terms, linear in ε, by their mean values over the period.

2. If the conditions of Theorem 1 are satisfied and the mean values of the periodic terms are zero, then the characteristic exponents do not contain terms of the first power in ε.

This result was obtained in a special case in [5].

Notes. 1. Theorem 1 remains valid for the problem with a small vector parameter ε.

2. The main merit of the formulation of Theorem 1 is the fact that the numbers λ_s do not depend on ε.

Another approach to investigating system (1.1) is also possible, based on the assumption that the mean value $\mathbf{X}_{10}(\varepsilon, \mathbf{x})$ of the function $\mathbf{X}_1(\varepsilon, \mathbf{x}, t)$ over the period is zero. In this case the λ_s are found as the roots of the characteristic equation of the system

$$\mathbf{x}^{\cdot} = \mathbf{X}_0(\mathbf{x}) + \varepsilon\mathbf{X}_{10}(\varepsilon, \mathbf{x})$$

and, in general, they depend on ε. Theorem 1 also holds in this case, of course. When this approach is used it is convenient to introduce one more small parameter μ and to examine the system

$$\mathbf{x}^{\cdot} = \mathbf{X}_0(x) + \mu\mathbf{X}_{10}(\mu, \mathbf{x}) + \varepsilon[\mathbf{X}_1(\varepsilon, \mathbf{x}, t) - \mathbf{X}_{10}(\varepsilon, \mathbf{x})]$$

with $\mu = \varepsilon$. Bearing this in mind, we will put $\mathbf{X}_{10}(\varepsilon, \mathbf{x}) \equiv \mathbf{0}$ below to simplify the formulations.

Resonance $2\lambda_1 = pi$. In this case the normal form of the linear system has the form

$$\begin{aligned}
\xi_j^{\cdot} &= i\varepsilon a_j(\varepsilon)(\eta_1 e^{-\lambda_1 t} - \bar{\eta}_1 e^{\lambda_1 t}) + \ldots \quad (j = 1, \ldots, m)\\
\eta_1^{\cdot} &= \lambda_1\eta_1 + i\varepsilon b(\varepsilon)e^{2\lambda_1 t}\bar{\eta}_1 + \ldots\\
\bar{\eta}_1^{\cdot} &= -\lambda_1\bar{\eta}_1 - i\varepsilon b(\varepsilon)e^{2\lambda_1 t}\eta_1 + \ldots
\end{aligned}$$

where all a_j equal zero when $\lambda_1 i$ is not an integer.

Making the replacement

$$\eta_1 = w_1\exp(\lambda_1 t), \quad \bar{\eta}_1 = \bar{w}_1\exp(-\lambda_1 t)$$

we obtain

$$\begin{aligned}
\xi_j^{\cdot} &= i\varepsilon a_j(\varepsilon)(w_1 - \bar{w}_1) + \ldots \quad (j = 1, \ldots, m)\\
w_j^{\cdot} &= i\varepsilon b(\varepsilon)\bar{w}_1 + \ldots, \quad \bar{w}_1^{\cdot} = -i\varepsilon b(\varepsilon)w_1 + \ldots
\end{aligned}$$

and the characteristic exponents have non-zero real parts $\kappa_{1,2} = \pm\varepsilon b(\varepsilon)$.

Theorem 2. Parametric resonance $2\lambda_1 = pi$, $p \in \mathbf{Z}$ almost always ($b \neq 0$) leads to instability, according to the first approximation.

Resonance $\lambda_1+\lambda_2=\mathrm{p}i$. In the variables w_1 and w_2 we have

$$w_1^{\cdot}=i\varepsilon b_1\overline{w}_2+\ldots,\quad w_2^{\cdot}=i\varepsilon b_2\overline{w}_1+\ldots \tag{1.5}$$

and the characteristic exponents of this system are $\kappa_{1,2}=\kappa_{3,4}=\pm\varepsilon\sqrt{(b_1b_2)}$. Hence the system is unstable according to the first approximation if $b_1b_2>0$.

Suppose $b_1b_2<0$. Then all k are pure imaginary and the characteristic exponents of the original system are

$$\lambda_1^*=\lambda_1+\varepsilon\sqrt{b_1b_2},\quad -\lambda_1^*,\quad \lambda_2^*=\lambda_2-\varepsilon\sqrt{b_1b_2},\quad -\lambda_2^*$$

In fact, if original system (1.3) has the solution $w_1=c_1\exp(\kappa t)\,(c_1=\mathrm{const})$ then

$$w_1^{\cdot}=\kappa c_1\exp(\kappa t)=i\varepsilon b_1\overline{w}_2,\quad w_2=c_2\exp(-\kappa t)\quad (c_2=\mathrm{const})$$

The calculated values of λ_1^* and λ_2^* show that resonance $\lambda_1^*+\lambda_2^*=pi$ occurs in the non-linear system.

Theorem 3. If $b_1b_2>0$ then the parametric resonance $\lambda_1+\lambda_2=pi$, $p\in\mathbf{Z}$, $p\neq 0$ leads to instability according to the first approximation. When $b_1b_2<0$ the characteristic exponents $\pm\lambda_1^*$, $\pm\lambda_2^*$ are purely imaginary and the resonance $\lambda_1^*+\lambda_2^*=pi$ occurs in the non-linear system.

The non-linear system. If there are no resonances

$$p_1\lambda_1+\ldots+p_n\lambda_n+ip=0;\quad p_1,\ldots,p_n,p\in\mathbf{Z} \tag{1.6}$$

system (1.2) is formally stable. This can be shown by adding the equation $\dot{\varepsilon}=0$ to (1.2) and normalizing the reversible system obtained with $m+1$ zero and n pairs of purely imaginary roots. As a result, we obtain the special case of [3]. Hence it follows that the other properties established in [3] for the non-resonance case are also true.

Suppose the resonance (1.6) with $p=0$ now occurs in the system. The resonance terms in normal form do not depend on time explicitly, and therefore the properties established for the autonomous system (with $\varepsilon=0$) in [1] remain valid for the quasi-autonomous system.

In the case of resonance (1.6) with $p\neq 0$ the resonance coefficients in normal form depend on ε, vanishing when $\varepsilon=0$. Hence we have the following [3]: third-order resonance, as a rule leads to instability, and stability in any finite order occurs in the case of fourth-order resonances. These conclusions were previously obtained for multifrequency resonances in [5].

Theorem 2 solves the problem of the stability in the case of the single-frequency second-order resonance. In the degenerate case $(b=0)$ the system is stable in any finite order as follows from [3] and from the above discussion.

Resonance $\lambda_1+\lambda_2=\mathrm{p}i$ *in a non-linear system.* Let the condition $b_1b_2<0$ be satisfied in Theorem 3. The characteristic exponents are then purely imaginary and the normal form of the system of the first non-linear approximation has the form

$$\begin{aligned}
\eta_1^{\cdot}&=\lambda_1^*\eta_1+i(A_{11}|\eta_1|^2+A_{12}|\eta_2|^2)\eta_1+i\varepsilon(B_{11}|\eta_1|^2+B_{12}|\eta_2|^2)\overline{\eta}_2e^{(\lambda_1+\lambda_2)t}+\\
&+i\varepsilon(C_{11}\overline{\eta}_1\overline{\eta}_2^2e^{2(\lambda_1+\lambda_2)t}+C_{12}\eta_1^2\eta_2e^{-(\lambda_1+\lambda_2)t})+\ldots\\
\eta_2^{\cdot}&=\lambda_2^*\eta_2+i\varepsilon(A_{21}|\eta_1|^2+A_{22}|\eta_2|^2)\overline{\eta}_1e^{(\lambda_1+\lambda_2)t}+i(B_{21}|\eta_1|^2+B_{22}|\eta_2|^2)\eta_2+\\
&+i\varepsilon(C_{21}\eta_1\eta_2^2e^{-(\lambda_1+\lambda_2)t}+C_{22}\overline{\eta}_1^2\overline{\eta}_2e^{2(\lambda_1+\lambda_2)t})+\ldots
\end{aligned}$$

where A_{jk}, B_{jk}, C_{jk} are certain constant coefficients and the complex-conjugate group of equations is omitted.

Putting

$$\eta_1 = w_1 \exp(\lambda_1^* t), \quad \eta_2 = w_2 \exp(\lambda_2^* t)$$

and changing to the variables

$$\rho_1 = w_1\overline{w}_1, \quad \rho_2 = w_2\overline{w}_2, \quad y = w_1w_2 + \overline{w}_1\overline{w}_2$$

we obtain the model system of the first non-linear approximation

$$z^{\cdot} = x\mathbf{Az}, \quad \mathbf{z} = (\rho_1, \rho_2, y)^T, \quad x = \pm\sqrt{4\rho_1\rho_2 - y^2} \tag{1.7}$$

$$\mathbf{A} = \|a_{jk}\|_1^3, \quad a_{11} = \varepsilon(B_{11} - C_{12}), \quad a_{12} = \varepsilon B_{12}, \quad a_{13} = \varepsilon C_{11}$$

$$a_{21} = \varepsilon A_{21}, \quad a_{22} = \varepsilon(A_{22} - C_{21}), \quad a_{23} = \varepsilon C_{22}$$

$$a_{31} = -B_{21} - A_{11} + \varepsilon C_{22}, \quad a_{32} = -B_{22} - A_{12} + \varepsilon C_{11}, \quad a_{33} = -\varepsilon(C_{11} + C_{22})$$

which was examined in [6]. As was established in [6], the necessary and sufficient condition of stability (1.7) in the structurally stable case is the requirement that the equation

$$G_3k^3 + G_2k^2 + G_1k + G_0 = 0 \tag{1.8}$$

should not have a positive solution k satisfying the condition

$$|B_{12}k^2 + (C_{21} - A_{22} + B_{11} - C_{12})k - A_{21}| < 2\sqrt{k}|C_{22} - kC_{11}| \tag{1.9}$$

The coefficients of Eq. (1.8) are calculated from the formulae

$$\varepsilon^{-2}G_3 = C_{11}^2(\varepsilon C_{11} - \alpha) + \varepsilon B_{12}(\mu - B_{12}C_{22})$$

$$\varepsilon^{-2}G_2 = C_{11}^2(\varepsilon C_{22} - \beta) + 2C_{11}C_{22}(\alpha - \varepsilon C_{11}) + \varepsilon B_{12}(C_{11}A_{21} - \delta) + \varepsilon\gamma(\mu - B_{12}C_{22})$$

$$\varepsilon^{-2}G_1 = 2C_{11}C_{22}\beta - C_{22}^2\alpha - \varepsilon C_{11}C_{22}^2 - \varepsilon A_{21}(\mu - B_{11}C_{22}) + \varepsilon\gamma(C_{11}A_{21} - \gamma)$$

$$\varepsilon^{-2}G_0 = C_{22}^2(\varepsilon C_{22} - \beta) + \varepsilon A_{21}(\nu - C_{11}A_{21})$$

$$\alpha = A_{12} + B_{22}, \quad \beta = A_{11} + B_{21}, \quad \gamma = B_{11} - C_{12} - A_{22} + C_{21}$$

$$\delta = (B_{11} + C_{21})C_{21}, \quad \nu = (C_{21} + B_{11})C_{22}, \quad \mu = (C_{12} + A_{22})C_{11}$$

Equation (1.8), apart from terms of order ε, can therefore be written in the form

$$(\alpha k + \beta)(C_{11}k - C_{22})^2 = \varepsilon(\ldots) \tag{1.10}$$

Hence it follows that, if $\alpha\beta < 0$, Eq. (1.8) always has a positive root calculated from the formula

$$k = -\beta/\alpha + \varepsilon(\ldots)$$

When the additional condition $C_{11}C_{22} > 0$ is satisfied Eq. (1.10) can also have other positive roots which, as can be easily shown, do not satisfy condition (1.9).

Thus, confining ourselves by the structurally stable situation, as in [6], we arrive at the following assertion.

Theorem 4. Let the boundary $\partial\mathbf{K}$ of the cone

$$\mathbf{K} = \{\rho_1, \rho_2, y\colon\ 4\rho_1\rho_2 \geqslant y^2, \ \rho_1 \geqslant 0, \ \rho_2 \geqslant 0\}$$

have no degenerate invariant rays. The model system is then stable if and only if at least one of the inequalities

$$\alpha\beta<0, \quad |A_{21}\alpha^2+\gamma\alpha\beta-B_{12}\beta^2|<2\sqrt{-\alpha\beta}|C_{22}\alpha+C_{11}\beta| \tag{1.11}$$

is not true.

When condition (1.11) is satisfied the original system (1.2) is unstable.

2. THE CONDITIONALLY PERIODIC SYSTEM

Now let the vector-function $\mathbf{X}_1(\varepsilon, \mathbf{x}, t)$ in (1.1) be conditionally periodic in t with the basis of the frequencies $\mathbf{\Omega}=(\omega_1, \ldots, \omega_k)$ while the vectors $\mathbf{\Lambda}$ and $\mathbf{\Omega}$ satisfy the condition of reducibility of the system [2, p. 92] to normal form. In this case the resonance terms are found from the equation

$$\begin{gathered}(q_1-r_1)\lambda_1+\ldots+(q_n-r_n)\lambda_n+i\langle \mathbf{P}, \mathbf{\Lambda}\rangle=0 \\ \langle \mathbf{P}, \mathbf{\Lambda}\rangle=p_1\omega_1+\ldots+p_n\omega_n\end{gathered} \tag{2.1}$$

and the dependence on time is given by an integer vector $\mathbf{P}$ in the form of the factor $i\langle\mathbf{P}, \ \mathbf{\Omega}\rangle t$ instead of $\exp(ipt)$ in the periodic case. Therefore, if

$$\lambda_s\pm\lambda_j+i\langle \mathbf{P}, \mathbf{\Omega}\rangle\neq 0 \tag{2.2}$$

system (1.3) reduces to the same normal form as in the periodic case and Theorem 1 remains valid.

When the resonance cases (Theorems 2–4) were considered we substituted the roots λ_s for the quantity ip. In the case of the conditionally periodic system this substitution also occurs but it is necessary to use $i\langle\mathbf{P}, \ \mathbf{\Omega}\rangle$ instead of ip. All the remaining discussions are as before, and, Theorems 2–4 therefore remain valid if to substitute $i\langle\mathbf{P}, \ \mathbf{\Omega}\rangle$ for the quantity ip in their formulations.

3. SYSTEMS CLOSE TO RESONANCE

Let us consider a reversible system (autonomous, periodic, and conditionally periodic) whose right-hand sides depend on a certain parameter ε. Let the system of the linear approximation be stable and reducible to an autonomous one with m zero and n pairs of pure imaginary roots $\pm\lambda_s(\varepsilon)\,(s=1, \ldots, n)$ and, when $\varepsilon=0$, let resonance occur in the system. We then have the non-resonance case when $\varepsilon\neq 0$, and the system is formally stable (for a conditionally periodic system this is true under the additional conditions of reducibility to normal form [2]). It is of interest to clarify, when $\varepsilon\neq 0$, what happens with the solutions which showed instability in the presence of a resonance, i.e. when $\varepsilon=0$. We shall consider this problem using the example of two characteristic cases of third-order and fourth-order resonances.

As above, we regard ε as the local variable and normalize the resonance system obtained with $\varepsilon\neq 0$. The model system is actually written out in [1] and up to $O(\varepsilon^2)$ and fourth-order infinitesimal terms in the phase variables it has the form

$$\begin{gathered}r_\alpha^{\cdot}=2B_\alpha\sin\theta\prod_{j=1}^{n} r_j^{p_j/2} \quad (\alpha=1,\ldots,n) \\ \theta^{\cdot}=a\varepsilon+\prod_{j=1}^{n} A_j r_j+\sum_{j=1}^{n} p_j B_j\prod_{k=1}^{n} r_k^{p_k/2-\delta_{jk}}\cos\theta\end{gathered} \tag{3.1}$$

$$p_1\lambda_1(0)+\ldots+p_n\lambda_n(0)=0,\quad p=\sum_{j=1}^{n} p_j=3 \text{ or } 4$$

where p_j are the natural numbers, a, A_j, and B_α are certain constants, but all $A_j=0$ when there is third-order resonance. For simplicity, we assume that there are no zero roots and all the frequencies are resonance frequencies.

In the non-degenerate cases, when there are no zeros among B_α, the growing solutions in the form of the rays $r_\alpha=k_\alpha r$ ($k_\alpha=\text{const}$, $\alpha=1,\ldots,n$), which exist for all B_α of the same sign [1], indicate the instability of system (3.1). Suppose all $B_\alpha>0$. Then one can take $k_\alpha=B_\alpha$ and the equations for r and θ have the form

$$\begin{aligned} r^{\cdot} &= 2\Pi\sin\theta r^{p/2},\quad \Pi=\prod_{j=1}^{n} B_j^{p_j/2} \\ \theta^{\cdot} &= a\varepsilon+2br+p\Pi\cos\theta r^{p/2-1},\quad b=\sum_{j=1}^{n} A_jB_j \end{aligned} \tag{3.2}$$

The system obtained is canonical with Hamilton function

$$H=a\varepsilon r+(b+2\Pi\cos\theta)r^{p/2}$$

and is completely analysed in the phase plane.

The phase portrait of system (3.2) with $p=3$ (third-order resonance) is given in Fig. 1(a) for the case $a\varepsilon>0$. Thus, bifurcation of equilibrium occurs when $\varepsilon=0$ and, apart from the zero position of equilibrium, a saddle-point with coordinates $\theta_*=\pi$, $r_*=(a\varepsilon/3\Pi)^2$ appears. The bifurcation diagram is shown in Fig. 1(b) where centres are marked by light dots and saddles are marked by crosses. The equation of the separatrices has the form

$$(a\varepsilon+2\Pi\sqrt{r}\cos\theta)r=-(a\varepsilon)^3/(3\Pi)^2$$

In the case close to fourth-order resonance, the phase portrait has a different form depending on whether system (3.2) is stable when $\varepsilon=0$. If $|b|>2\Pi$ (the system is stable) then, when $a\varepsilon b>0$, all the trajectories are closed and surround zero. When $a\varepsilon b<0$ the phase portrait is given in Fig. 2(a). Two new steady saddle-points with coordinates $\theta_*\pi$, $r_*=-a\varepsilon/(2b-4\Pi)$; $\theta_*=0$, $r_*=-a\varepsilon/(2b+4\Pi)$ appear. In this case the equations of the separatrices have the form

$$[a\varepsilon+(b+2\Pi\cos\theta)r]r=(a\varepsilon)^2/(2b\pm 4\Pi)$$

The bifurcation diagram is shown in Fig. 2(b).

When $|b|<2\Pi$ (system (3.2) is unstable if $\varepsilon=0$) the phase portrait has the same form as in Fig. 1(a). When $|b|=2\Pi$ with $a\varepsilon b>0$ we have periodic motions. When $a\varepsilon b<0$ the phase portrait is the same as in Fig. 1(a), the saddle then appears in the right-hand half-plane (when $a\varepsilon>0$).

4. PARAMETRIC INSTABILITY IN THE PROBLEM OF THE ROLLING OF CHAPLYGIN'S SPHERE ALONG AN ABSOLUTELY ROUGH PLANE

Let us consider the motion of a heavy rigid body of spherical shape along a stationary rough horizontal plane. Let the body be Chaplygin's sphere [7], and suppose its geometrical centre coincides with the centre of gravity. Note this problem has been solved by Chaplygin in quadratures [8]. The stability of the rolling of the sphere along a straight line is investigated below.

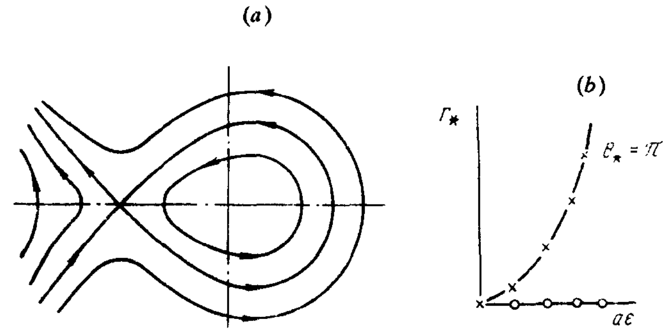

Fig. 1.

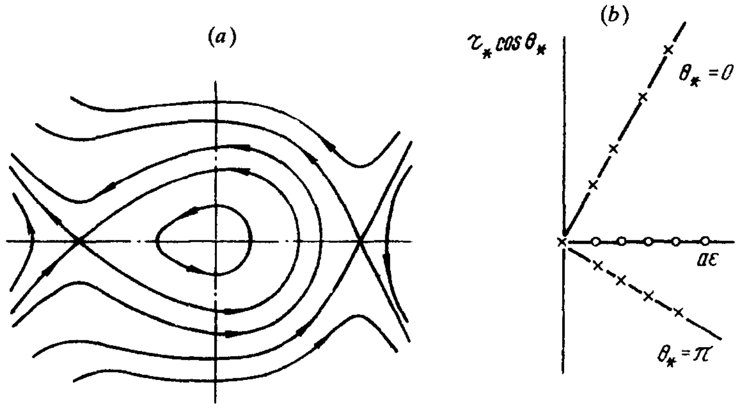

Fig. 2.

Let the sphere of radius R have mass m and let A, B and C be the moments of inertia relative to the moving system of coordinates specified by the principle central axes of inertia. We obtain the equations of motion of the sphere as a special case of the equations given in [9]

$$\begin{aligned}
&[A+mR^2(\gamma_2^2+\gamma_3^2)]\dot\omega_1 - mR^2\gamma_1\gamma_2\dot\omega_2 - mR^2\gamma_1\gamma_3\dot\omega_3 = (B-C)\omega_2\omega_3 \\
&[B+mR^2(\gamma_3^2+\gamma_1^2)]\dot\omega_2 - mR^2\gamma_2\gamma_3\dot\omega_3 - mR^2\gamma_2\gamma_1\dot\omega_1 = (C-A)\omega_3\omega_1 \qquad (4.1)\\
&[C+mR^2(\gamma_1^2+\gamma_2^2)]\dot\omega_3 - mR^2\gamma_3\gamma_1\dot\omega_1 - mR^2\gamma_2\gamma_3\dot\omega_2 = (A-B)\omega_1\omega_2 \\
&\dot\gamma_1+\gamma_3\omega_2-\gamma_2\omega_3=0, \quad \dot\gamma_2+\gamma_1\omega_3-\gamma_3\omega_1=0, \quad \dot\gamma_3+\gamma_2\omega_1-\gamma_1\omega_2=0 \\
&\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T, \qquad \boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)^T
\end{aligned}$$

where $\boldsymbol{\omega}$ is the angular velocity of rotation of the body, and $\boldsymbol{\gamma}$ is the unit vector at the point of contact of the sphere with the plane vertically upwards.

System (4.1) is reversible and is not changed when the signs of t and the variables in one of the pairs (ω_j, λ_j) or in the vector $\boldsymbol{\omega}$ change, i.e. the system has the four linear automorphisms.

Equations (4.1) have the partial solution

$$\omega_1 = \omega_2 = 0, \ \ \gamma_3 = 0, \ \ \omega_3 = \omega_*(\text{const}), \ \ \gamma_1 = \cos\omega_* t, \ \ \gamma_2 = \sin\omega_* t \qquad (4.2)$$

which describes the rolling of the sphere with constant angular velocity ω_* along a certain straight line. We set up a system of equations in variations in the neighbourhood of motion (4.2). This system is decomposed into two groups of equations, one of which gives

$$\omega_3 = \omega(\text{const}),\quad \gamma_1 = \cos\varphi,\quad \gamma_2 = \sin\varphi\quad (\varphi = \omega t)$$
$$\gamma_3^{\cdot} = -\omega_1 \sin\varphi + \omega_2 \cos\varphi$$

In the second group of equations

$$(A + mR^2\sin^2\varphi)\omega_1^{\cdot} - mR^2\sin\varphi\cos\varphi\omega_2^{\cdot} = (B - C)\omega\omega_2$$
$$(B + mR^2\cos^2\varphi)\omega_2^{\cdot} - mR^2\sin\varphi\cos\varphi\omega_1^{\cdot} = (C - A)\omega\omega_1$$

we change to the new variables

$$p = \omega_1\cos\varphi + \omega_2\sin\varphi,\quad q = \omega_1\sin\varphi - \omega_2\cos\varphi$$

and the time φ and solve the equations obtained for the derivatives. As a result, we obtain the reversible system

$$dp/d\varphi = -q + [a_*\sin 2\varphi p + (b_* + a_*\cos 2\varphi)q]/(2\Delta)$$
$$qq/d\varphi = p - [(c_* + e_*\cos 2\varphi)p + e_*\sin 2\varphi q]/(2\Delta) \tag{4.3}$$
$$a_* = (C - A)(A + mR^2) - (C - B)(B + mR^2),\quad b_* = (C - A)(A + mR^2) +$$
$$+ (C - B)(B + mR^2),$$
$$c_* = A(C - A) + B(C - B),\quad e_* = A(C - A) - B(C - B)$$
$$\Delta = -[AB + mR^2(A\cos^2\varphi + B\sin^2\varphi)]$$

When $A = B$ system (4.3) becomes autonomous and in this case the square of the frequency of the small oscillations equals

$$\Omega_*^2 = \frac{C}{A}\,\frac{C + mR^2}{A + mR^2}$$

which is identical with the result in [10] for the case of Chaplygin's sphere.

Consider the case when A is close to B: $B = A(1 + 2\varepsilon)$. We then have

$$a_* = 2A\varepsilon(2A - C + mR^2 + 2A\varepsilon),\qquad b_* = 2(C - A)(A + mR^2) - a_*$$
$$e_* = 2A\varepsilon(2A - C - 2A\varepsilon),\quad c_* = 2A(C - A) - e_*$$
$$\Delta = -[A(A + mR^2) + 2A\varepsilon(A + mR^2\sin^2\varphi]$$

and system (4.3) takes the form (the periodic terms are written up to terms of order ε)

$$dp/d\varphi = -\kappa_1 q - \alpha\varepsilon\sin 2\varphi p + \beta\varepsilon\cos 2\varphi q + \ldots$$
$$dq/d\varphi = \kappa_2 p + \alpha\varepsilon\cos 2\varphi p + \gamma\varepsilon\sin 2\varphi q + \ldots$$
$$\kappa_1 = 1 + \frac{1}{2\Delta_* f}\left(b_* + \frac{a_*\beta_*}{1+f}\right),\quad \kappa_2 = 1 + \frac{1}{2\Delta_* f}\left(c_* + \frac{e_*\beta_*}{1+f}\right) \tag{4.4}$$
$$\alpha = \frac{2A - C + mR^2}{A + mR^2},\quad \beta = \frac{A(2A - C) + CmR^2}{A(A + mR^2)},\quad \gamma = \frac{2A - C}{A + mR^2}$$
$$f = \sqrt{1 - \beta_*^2},\quad \beta_* = mR^2\varepsilon / \Delta_*,\quad \Delta_* = A + mR^2 + 2A\varepsilon + mR^2\varepsilon$$

Let us average this system over φ in the period 2π. It then follows from Theorem 1 that the linear system is stable provided that

$$\Omega^2 = \kappa_1\kappa_2 = \frac{C}{A}\,\frac{C + mR^2}{A + mR^2}(1 - 2\varepsilon) + \ldots \neq \frac{l^2}{4}, \quad l \in \mathbf{Z} \tag{4.5}$$

(Ω is the oscillation frequency of the averaged system), while its characteristic exponents, to terms of order ε, equal $\pm i\Omega$. Otherwise the second-order resonance $2\Omega = l$ occurs in the system. This resonance, as follows from the form of the right-hand sides of (4.4), can cause instability when $l = 2$. If the quantities A and C differ by terms of order ε and Chaplygin's sphere is nearly uniform.

Let the parameters be such that $\Omega = 1$. We change to complex-conjugate variables in (4.4)

$$\eta = \kappa_2 p + i\Omega q, \quad \bar{\eta} = \kappa_2 p - i\Omega q$$

and normalize the system obtained. As a result, we have a system of the form (4.4) in which

$$4b = \alpha(\kappa_2 + \Omega)/\kappa_2 + (\kappa_2\beta + \Omega\gamma)/\Omega$$

Now, taking into account that when $\Omega = 1$, up to terms of order ε

$$\alpha = 1 + \ldots, \quad \beta = 1 + \ldots, \quad \gamma = A/(A + mR^2), \quad \kappa_2 = 1 + \ldots$$

we calculate

$$4b = 3 + A/(A + mR^2) + \ldots \neq 0$$

By Theorem 2, the rolling of the sphere is unstable in this case.

Theorem 5. Let Chaplygin's sphere of radius R have mass m, and let A, $B = A(1 + 2\varepsilon)$ and C be the principal central moments of inertia. The rolling of the sphere along a straight line is formally stable when ε is sufficiently small and condition (4.5) is satisfied. When $\Omega = 1$ parametric resonance occurs and the rolling is unstable according to the first approximation.

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